# Math Boot Camp: Volume Elements

*You can skip this boot camp if you can answer the following questions:*

<u>Example</u>

To prove that the volume of a sphere of radius $R$ is $\frac{4}{3}\pi R^3$, we can compute the following integral,

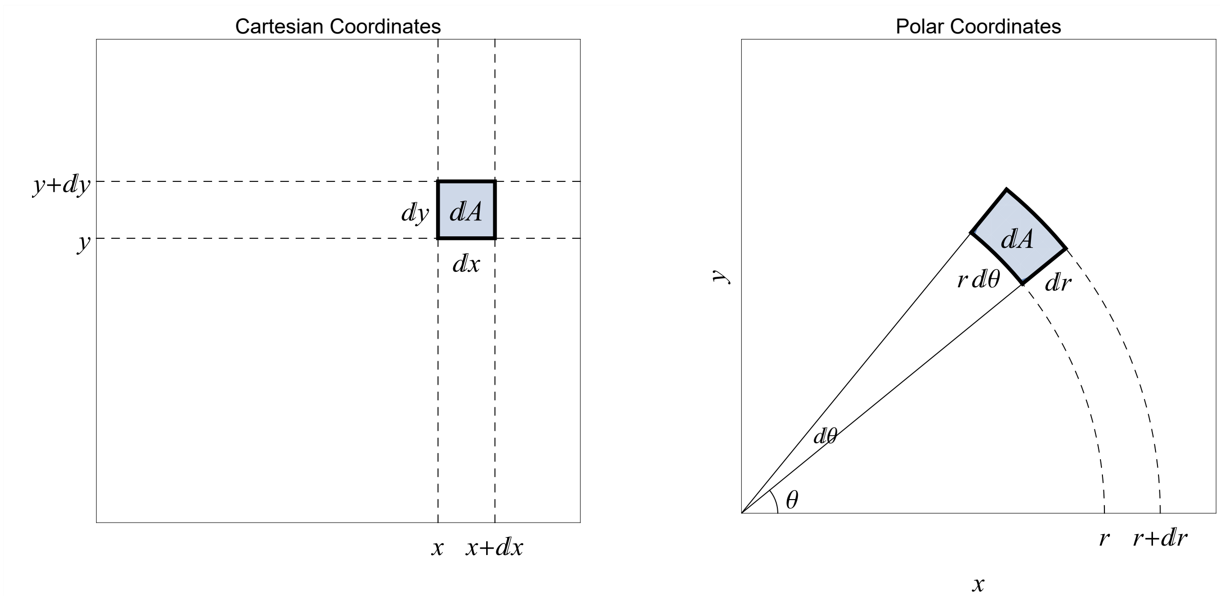
$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2 \operatorname{Sin}[\theta]\, dr\, d\theta\, d\phi \tag{1}$$

Why does the factor $r^2 \operatorname{Sin}[\theta]$ appear in the integrand?

<u>Example</u>

In the same spirit as the problem above, we want to prove that the surface area of a sphere of radius $R$ is $4\pi R^2$. Set up the appropriate integral, evaluate it, and ensure that you get the proper answer.

## 2D Coordinate Systems




### 2D Cartesian Coordinates

The basic question we want to answer is as follows. When computing 2D integrals in Cartesian coordinates, we are familiar with the area element $dx\, dy$, such as when we write $\int\int f[x, y]\, dx\, dy$. But where does $dx\, dy$ come from? We will first answer this question in Cartesian coordinates, and then use this same methodology to find the area and volume elements in other 2D and 3D coordinate systems.

If we consider a generic point $(x, y)$ and infinitesimally displace it in the $x$-direction and $y$-direction by $dx$ and $dy$, respectively, we find the points $(x, y)$, $(x + dx, y)$, $(x, y + dy)$, and $(x + dx, y + dy)$ as shown in the diagram above. What is the area of the region encompassed by these points? Simply $dx\, dy$!

$$\text{Cartesian coordinate area element} = r\, dr\, d\theta \tag{2}$$

The procedure is the same in every other coordinate system. You identify which variables you are going to vary and you infinitesimally vary them all one at a time. The area/volume of the resulting shape will be the desired infinitesimal area/volume element in that coordinate system.

### 2D Polar Coordinates

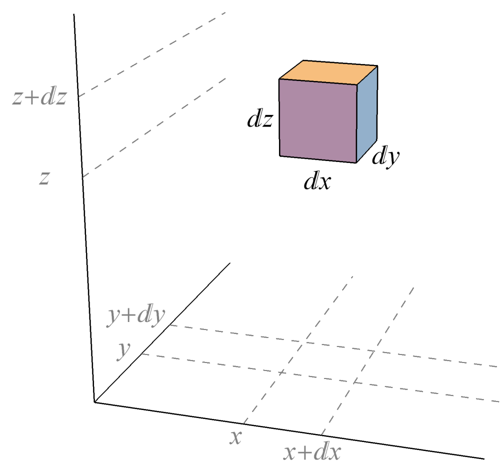
Consider a generic point $(r, \theta)$ in polar coordinates. As in the case of Cartesian coordinates, we will vary both of these coordinates to obtain the four points $(r, \theta)$, $(r + dr, \theta)$, $(r, \theta + d\theta)$, and $(r + dr, \theta + d\theta)$ as shown in the diagram above. In the limit when $d\theta$ and $dr$ are infinitesimals, the area element becomes a rectangle, and hence its area equals $(dr)\,(r\,d\theta) = r\,dr\,d\theta$. Thus,

$$\text{polar coordinate area element} = r\,dr\,d\theta \tag{3}$$

Note that the length of the inner curved segment cannot be $d\theta$, since both $\theta$ and $d\theta$ are angles and do not have the units of length. Both $dr$ and $r\,d\theta$ have the correct units of length.

## 3D Cartesian Coordinates

Now let's blast our way into 3D. As in 2D, we will begin with the simple Cartesian coordinate system. Given the coordinate $(x, y, z)$, we now infinitesimally vary each coordinate to obtain the eight new points $(x, y, z)$, $(x + dx, y, z)$, $(x, y + dy, z)$, $(x, y, z + dz)$, $(x + dx, y + dy, z)$, $(x + dx, y, z + dz)$, $(x, y + dy, z + dz)$, and $(x + dx, y + dy, z + dz)$. As you can guess, these form a cube with side lengths $dx$, $dy$, and $dz$.



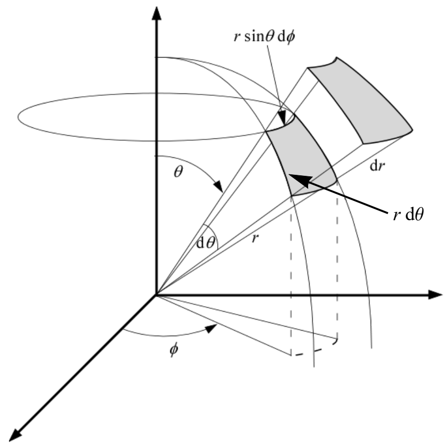

Therefore, the volume element in 3D Cartesian coordinates is

$$\text{Cartesian coordinate volume element} = dx\,dy\,dz \tag{4}$$

## 3D Spherical Coordinates

We now shift to spherical coordinates, we can compute the volume element by considering a generic point $(r, \phi, \theta)$ and looking at the volume element found by infinitesimally displacing its three coordinates by $dr$, $d\phi$, and $d\theta$. As shown in the diagram below, the volume of this element equals $(dr)\,(r\,d\theta)\,(r\,\mathrm{Sin}[\theta]\,d\phi) = r^2\,\mathrm{Sin}[\theta]\,dr\,d\theta\,d\phi$. Thus,

$$\text{spherical coordinate volume element} = r^2\,\mathrm{Sin}[\theta]\,dr\,d\theta\,d\phi \tag{5}$$




Example

To prove that the volume of a sphere of radius $R$ is $\frac{4}{3}\pi R^3$ by direct integration.

Solution

The volume of a solid is always given by $\int_{\text{solid}} dv$ where $dv$ is the infinitesimal volume element and the integration covers the entire solid. Working in spherical coordinates, we cover the sphere if we let $r \in [0, R]$, $\theta \in [0, \pi]$, and $\phi \in [0, 2\pi]$, using the spherical coordinate volume element $dv = r^2 \operatorname{Sin}[\theta]\, dr\, d\theta\, d\phi$,

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2 \operatorname{Sin}[\theta]\, dr\, d\theta\, d\phi = \left(\frac{1}{3}R^3\right)(2)(2\pi) = \frac{4}{3}\pi R^3 \tag{6}$$

as desired! □

## Surfaces in Spherical Coordinates

The diagram above also enables us to calculate the infinitesimal area elements when we integrate over only two of the spherical coordinates.

One common example is to integrate over the surface of a sphere. In that case, we only vary $\theta$ and $\phi$ by $d\theta$ and $d\phi$, respectively, and we do not vary $r$ at all. The corresponding area element is given in the diagram above as

$$\text{spherical coordinate area element} = r^2 \operatorname{Sin}[\theta]\, d\theta\, d\phi \qquad (\text{integrating } \theta \text{ and } \phi) \tag{7}$$

Example

Prove that the surface area of a sphere of radius $R$ is $4\pi R^2$ by direct integration.

Solution

We integrate over the entire sphere by letting $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$ while using the spherical coordinate area element $R^2 \operatorname{Sin}[\theta]\, d\theta\, d\phi$,

$$\int_0^{2\pi}\int_0^{\pi} R^2 \operatorname{Sin}[\theta]\, d\theta\, d\phi = R^2(2)(2\pi) = 4\pi R^2 \tag{8}$$

as desired! □

Another important check is to ensure that integration in spherical coordinates reduces to integration in polar coordinates when $\theta \to \frac{\pi}{2}$ and then consider $\phi$ as our polar coordinate. Looking back at the diagram above, we find the product of the infinitesimal lengths associated with the $r$ coordinate (which is $dr$) and the $\phi$ coordinate (which is $r \operatorname{Sin}[\theta]\, d\phi = r\, d\phi$ when $\theta = \frac{\pi}{2}$) will result in the area element

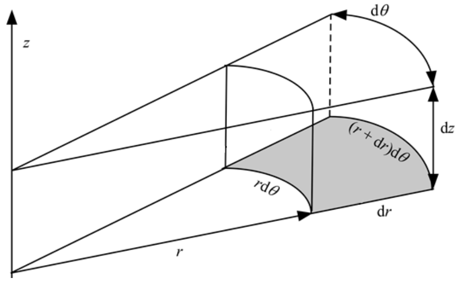
$$\text{spherical coordinate area element} = r\, dr\, d\phi \qquad (\text{integrating } r \text{ and } \phi) \tag{9}$$

As expected, this is identical to the polar coordinate area element Equation (3), aside from the change in the definition of the polar angle.

## 3D Cylindrical Coordinates

As we did for spherical coordinates, we displace a generic point $(r, \theta, z)$ in cylindrical coordinates using $dr$, $d\theta$, and $dz$. As shown in the diagram below, the volume of this element equals $(dr)(r\,d\theta)(dz) = r\,dr\,d\theta\,dz$. Thus,

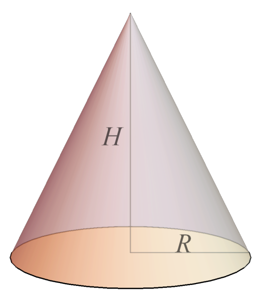
$$\text{cylindrical coordinate volume element} = r\,dr\,d\theta\,dz \tag{10}$$




You typically want to match your coordinate system to the geometry of the problem. Hence, you tend to you spherical coordinates when working with spheres and cylindrical coordinates in problems with rotational symmetry about one axis, such as in this next problem.

<u>Example</u>

Compute the volume of a cone with base radius $R$ and height $H$ using direct integration with cylindrical coordinates.




<u>Solution</u>

Taking the volume integral of the constant function 1 times the volume element $r\,dr\,d\theta\,dz$,

$$\begin{aligned} V &= \int_0^H \int_0^{2\pi} \int_0^{\left(1-\frac{z}{H}\right)R} r\,dr\,d\theta\,dz \\ &= \int_0^H \int_0^{2\pi} \frac{1}{2}\left(1-\frac{z}{h}\right)^2 R^2\,d\theta\,dz \\ &= \int_0^H \pi\left(1-\frac{z}{H}\right)^2 R^2\,dz \\ &= \frac{1}{3}\pi R^2 H^3 \end{aligned} \tag{11}$$

In the Integration Bootcamp, we carry out this same integral using Cartesian coordinates, which is much more painful. In cylindrical coordinates, this integral is *significantly easier to both setup and evaluate*. That is the reason why we use different coordinate systems: cylindrical coordinates let you utilize the symmetry of the problem.

Of course, we could have chosen to go wild and choose the $z$-axis to lie off the axis of the cone; the integral would still yield the same answer, but the calculation would be significantly tougher. Many problems in physics are ultimately solved by approaching the problem from the right perspective. □