

Math Boot Camp: Unit Vectors in Different Coordinate Systems

You can skip this boot camp if you can answer the following question:

Example

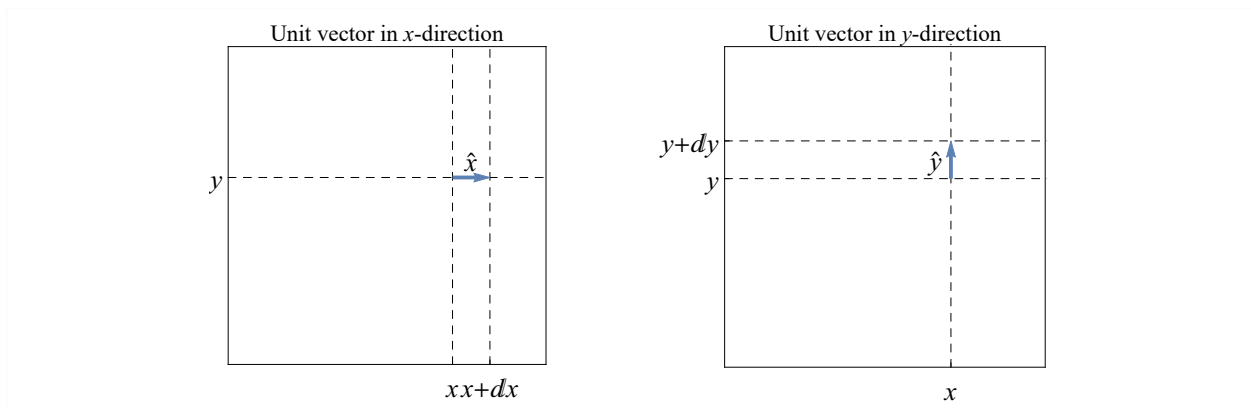
Write the polar unit vectors \hat{r} and $\hat{\theta}$ in terms of the Cartesian unit vectors \hat{x} and \hat{y} .

Unit Vectors

We are familiar with the unit vectors in Cartesian coordinates, where \hat{x} points in the x -direction and \hat{y} points in the y -direction. Here, we will first state the general definition of a unit vector, and then extend this definition into 2D polar coordinates and 3D spherical coordinates.

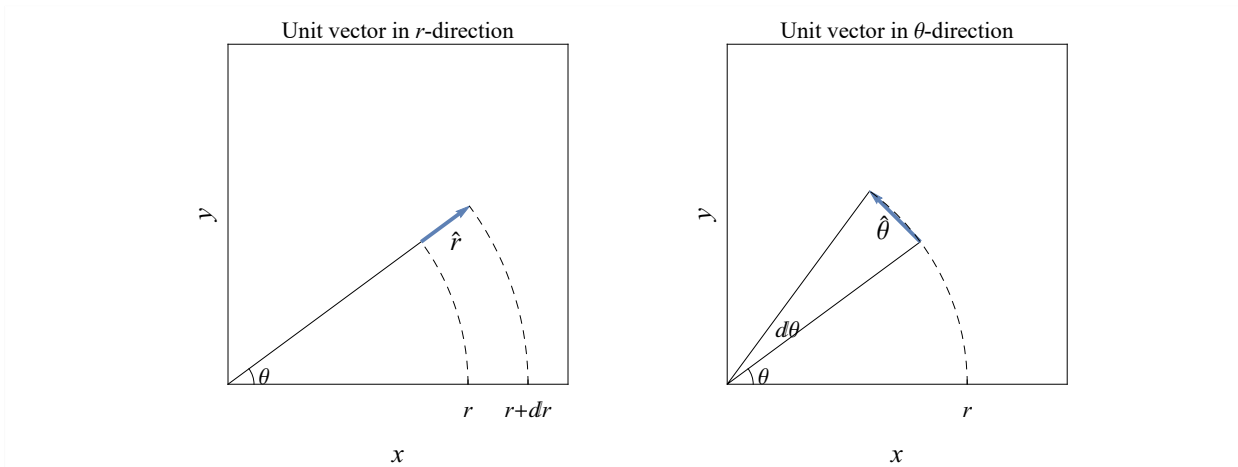
2D Cartesian Coordinates

Consider a point (x, y) . The unit vector of the first coordinate x is defined as the vector of length 1 which points in the direction from (x, y) to $(x + dx, y)$. The vector \hat{x} is shown below (left) and is seen to lie along the x -direction, as expected. Similarly \hat{y} is defined as the unit vector pointing from (x, y) to $(x, y + dy)$ as shown below (right).



2D Polar Coordinates

We follow this same procedure in polar coordinates to find \hat{r} and $\hat{\theta}$. Starting from the point (r, θ) , the \hat{r} unit vector points lies in the direction from (r, θ) to $(r + dr, \theta)$, which means that it points directly outwards from the origin. Similarly, $\hat{\theta}$ is a unit vector pointing from (r, θ) to $(r, \theta + d\theta)$, which implies that $\hat{\theta}$ points counter-clockwise from (r, θ) along the circle of radius r . Note that in the definition of $\hat{\theta}$, it is critical that $d\theta$ is infinitesimally small, whereas in the definition of \hat{r} and the definition of the Cartesian unit vectors this was not important.



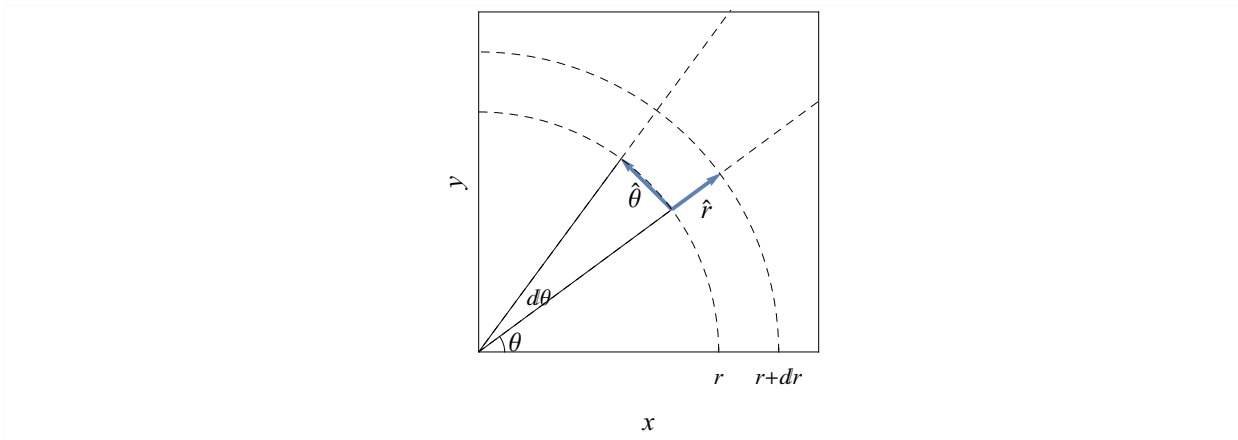
Unlike the \hat{x} and \hat{y} Cartesian coordinate unit vectors, the polar unit vectors \hat{r} and $\hat{\theta}$ depend upon the point (r, θ) , although you can quickly show that they only depend upon θ and not upon r . Here are some examples:

- At $(r, \theta) = (r, 0)$: $\hat{r} = \hat{x}$ and $\hat{\theta} = \hat{y}$
- At $(r, \theta) = (r, \frac{\pi}{2})$: $\hat{r} = \hat{y}$ and $\hat{\theta} = -\hat{x}$
- At $(r, \theta) = (r, \frac{\pi}{4})$: $\hat{r} = \frac{\hat{x} + \hat{y}}{\sqrt{2}}$ and $\hat{\theta} = \frac{\hat{x} - \hat{y}}{\sqrt{2}}$

Now let's find the general relations between the polar and Cartesian unit vectors.

Example

Write the polar unit vectors \hat{r} and $\hat{\theta}$ in terms of the Cartesian unit vectors \hat{x} and \hat{y} .



Solution

The geometrical approach is easiest. Since \hat{r} points along the line containing the origin and the point (r, θ) , its components must be

$$\hat{r} = \text{Cos}[\theta] \hat{x} + \text{Sin}[\theta] \hat{y} \tag{1}$$

$\hat{\theta}$ points 90° counter-clockwise of \hat{r} , which implies

$$\hat{\theta} = -\text{Sin}[\theta] \hat{x} + \text{Cos}[\theta] \hat{y} \tag{2}$$

As a double check, note that both \hat{r} and $\hat{\theta}$ are unit vectors (i.e. have norm 1), the two unit vectors are perpendicular ($\hat{r} \cdot \hat{\theta} = 0$), and in the limit $\theta = 0$ we regain $\hat{r} = \hat{x}$ and $\hat{\theta} = \hat{y}$, so that $\hat{\theta}$ points in the correct direction relative to \hat{r} . □

Using polar coordinates will greatly simplify problems that have circular geometries (for example, a particle undergoing uniform circular motion). However, when using any coordinate system other than Cartesian coordi-

nates, you must remember that the unit vectors \hat{r} and $\hat{\theta}$ can vary in time (if the base point about which they are computed changes in time). To find how these unit vectors vary with time, we differentiate Equations (1) and (2) with respect to time, using the chain rule and the fact that for Cartesian coordinates $\frac{dx}{dt} = \frac{dy}{dt} = 0$ irrespective of the base point,

$$\frac{d\hat{r}}{dt} = -\dot{\theta} \sin[\theta] \hat{x} + \dot{\theta} \cos[\theta] \hat{y} = \dot{\theta} \hat{\theta} \quad (3)$$

$$\frac{d\hat{\theta}}{dt} = -\dot{\theta} \cos[\theta] \hat{x} - \dot{\theta} \sin[\theta] \hat{y} = -\dot{\theta} \hat{r} \quad (4)$$

Fantastic! Now we are ready to calculate \vec{v} . The position vector using polar unit vectors has the very simple form

$$\vec{r} = r \hat{r} \quad (5)$$

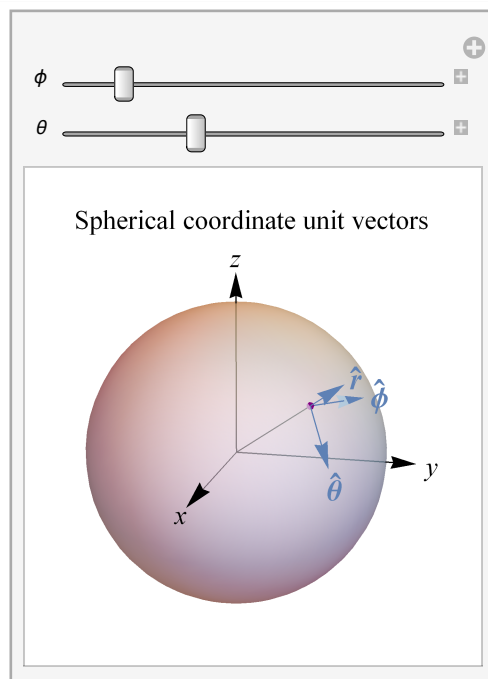
Differentiating with respect to time, we obtain the velocity

$$\begin{aligned} \vec{v} &= \frac{d\vec{r}}{dt} \\ &= \dot{r} \hat{r} + r \frac{d\hat{r}}{dt} \\ &= \dot{r} \hat{r} + r \dot{\theta} \hat{\theta} \end{aligned} \quad (6)$$

In other words, $\vec{v} = v_r \hat{r} + v_\theta \hat{\theta}$ where $v_r = \dot{r}$ is the velocity in the radial direction and $v_\theta = r \dot{\theta}$ is the velocity in the $\hat{\theta}$ direction.

3D Spherical Coordinates

We extend procedure above to determine the spherical coordinate unit vectors \hat{r} , $\hat{\phi}$, and $\hat{\theta}$. As shown below, \hat{r} points radially outward, $\hat{\phi}$ points in the x - y plane along the polar angle, and $\hat{\theta}$ points in the azimuthal direction.



Note that the unit vectors are all orthogonal to each other, and that the coordinate system is right handed. In the language of the cross product, this implies that $\hat{r} \times \hat{\theta} = \hat{\phi}$ (similar to how in Cartesian coordinates $\hat{x} \times \hat{y} = \hat{z}$).

As a fun exercise, you can show that the spherical coordinate unit vectors are related to the Cartesian coordinate vectors by

$$\begin{aligned}
 \hat{r} &= \sin[\theta] \cos[\phi] \hat{x} + \sin[\theta] \sin[\phi] \hat{y} + \cos[\theta] \hat{z} \\
 \hat{\theta} &= \cos[\theta] \cos[\phi] \hat{x} + \cos[\theta] \sin[\phi] \hat{y} - \sin[\theta] \hat{z} \\
 \hat{\phi} &= -\sin[\phi] \hat{x} + \cos[\phi] \hat{y}
 \end{aligned}
 \tag{7}$$

You will find in Ph 1b that descriptions of spherically symmetric setups are *much* simpler in terms of these unit vectors. For example, we will see next term that the Coulomb force from a point charge q exerts an electric field E with magnitude

$$E = \frac{kq}{r^2} \tag{8}$$

which points in the radial direction. In Cartesian coordinates, we would express this relation as

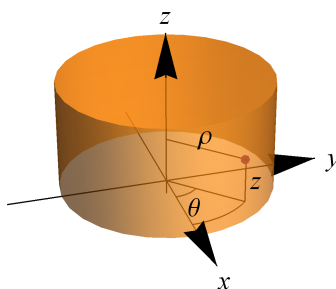
$$\vec{E} = \frac{kq}{x^2+y^2+z^2} \frac{x\hat{x}+y\hat{y}+z\hat{z}}{(x^2+y^2+z^2)^{1/2}} \tag{9}$$

In contrast, in spherical coordinates this equation takes the more compact form

$$\vec{E} = \frac{kq}{r^2} \hat{r} \tag{10}$$

3D Cylindrical Coordinates

When problems have cylindrical symmetry, it is often easiest to work in cylindrical coordinates. Given a point (ρ, θ, z) shown in the diagram below, what are the cylindrical coordinate vectors $\hat{\rho}$, $\hat{\theta}$, and \hat{z} in terms of the Cartesian coordinate unit vectors?



This question is straightforward if we realize that cylindrical coordinates are equivalent to 2D polar coordinates with the Cartesian z -coordinate tacked on. This implies that the cylindrical coordinate unit vectors are given by Equations (1) and (2) as

$$\hat{\rho} = \cos[\theta] \hat{x} + \sin[\theta] \hat{y} \tag{11}$$

$$\hat{\theta} = -\sin[\theta] \hat{x} + \cos[\theta] \hat{y} \tag{12}$$

Meanwhile, the z -coordinate is identical to the Cartesian z -coordinate, so its corresponding unit vector is trivially

$$\hat{z} = \hat{z} \tag{13}$$

irrespective of the base point (ρ, θ, z) .

Mathematica Initialization